

LIFETIME OF A SYMMETRICALLY PULSATING BUBBLE

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This article examines the problem of the time until the rupture of the surface of a gas bubble. The bubble initially undergoes spherically symmetric pulsations in a fluid which is quiescent at infinity. We construct a model of the rupture of the smooth surface of the pulsating bubble due to the growth of initial perturbations caused by thermal fluctuations. The source of the perturbations is of the minimum possible amplitude. It is shown that the thermal fluctuations are sufficient for rapid rupture of the bubble surface at moderate as well as high pressures. The dependence of the number of cycles to rupture on the pressure drop is determined, examples are presented for gas bubbles, and an estimate is made for vapor bubbles.

1. Dynamics of Perturbations Due to Thermal Fluctuations. The deviation of the radius r of the surface of a bubble from the radius of a sphere $R'(t)$ can be represented by a series in associated Legendre functions with random coefficients $a_{nm}(t)$, $b_{nm}(t)$:

$$\zeta = r - R' = \sum_{n=0}^{\infty} \sum_{m=0}^n (a_{nm} P_n^m \cos m\varphi + b_{nm} P_n^m \sin m\varphi). \quad (1.1)$$

Since the amplitudes of two different waves are statistically independent and since $\overline{a_{nm}^2} = \overline{b_{nm}^2}$, we obtain the following from Eq. (1.1) for the mean square deviation of the surface

$$\overline{\zeta^2} = \sum \sum \overline{a_{nm}^2} (P_n^m(\cos \theta))^2. \quad (1.2)$$

We assume that the mean square amplitude of the wave at the initial moment $t = 0$ coincides with the corresponding equilibrium value $(a_{nm}^2)_s$, due to thermal fluctuations:

$$t = 0, \quad \overline{a_{nm}^2} = (a_{nm}^2)_s. \quad (1.3)$$

The thermal fluctuations may be the main source of initial perturbations in the short-wave region — especially for bubbles in two-phase media. Using the physical concept (see [1], for example) of equilibrium thermal fluctuations in distributed systems — which was first applied to a capillary wave on the surface of a liquid in [2] — we write the mean potential energy of an individual capillary wave U in the equilibrium state as

$$U = (1/2)k'T'$$

(k' is the Boltzmann constant; T' is absolute temperature). Considering that U is proportional to the square of the amplitude of the perturbation of the bubble surface, we obtain the following for a wave with the amplitude a_{nm}

$$\sigma' \gamma(m) \frac{\pi n (n+m)!}{2 (n-m)!} (\overline{a_{nm}^2})_s = \frac{1}{2} k'T', \quad (1.4)$$

$$n \gg 1; \gamma = 2, m = 0; \gamma = 1, m > 0.$$

Allowing for the fact that the law of change in amplitude is independent of m [3, 4], it follows from (1.3) and (1.4) that for all $t \geq 0$

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$$\overline{a_{nm}^2} = 2 \frac{(n-m)!}{(n+m)!} \overline{a_{n0}^2} \quad (1.5)$$

By virtue of the summation theorem for associated Legendre functions

$$(P_n)^2 + 2 \sum_{m=1}^n \frac{(n-m)!}{(n+m)!} (P_n^m)^2 = 1$$

In accordance with the homogeneity of the problem with respect to the angle θ , we obtain the following from (1.2-1.5) for short waves on the bubble

$$\overline{\xi^2} = \sum_n \overline{a_n^2}, \quad (\overline{a_n^2})_S = \frac{k'T'}{2\pi n\sigma} \quad (1.6)$$

The logarithmic divergence of sum (1.6) at $n \rightarrow \infty$ is unimportant, since minimum wavelength is restricted to the molecular scale.

The dependence of $\overline{a_n^2}$ on time is conveniently represented in discrete form for two points of the cycle $t = (1/2)T$, $T - t_* - \delta$ (T is the period of pulsation). Asymptotic solution (1.5) [4] is valid when δ is small. We assign to each point a half-cycle number N in sequential order. Thus, in accordance with (1.3-1.5), we have the following for the indicated points [4]

$$\begin{aligned} \overline{a_{nm}^2} &= (\overline{a_{nm}^2})_S \exp F(N) (-q(0)/q(R))^{1/2} / R^3, \\ F &= 2\mu_1 T [N - 0] + 4H(N - [N - 0]), \quad 2N = 1, 2, \dots, \\ q &= -R / R + \sigma n^2 / R^3, \quad \exp(\mu_1 T) = 2 \exp(2H) \cos(2\sqrt{n} I_0), \\ I_0 &= \int_0^{t_*} \sqrt{q} dt, \quad q(t_*) = q(T - t_*) = 0 \end{aligned} \quad (1.7)$$

([] denotes the integral part of the number). To calculate the sum (1.6) determined by (1.7), we need to evaluate the behavior of H in the neighborhood of the index n that ensures $\max H$. The below equation is valid at point $\max H$

$$\frac{d^2 H}{dn^2} = -2c \frac{K_0}{n^{3/2}}, \quad H = \sqrt{n} K_0, \quad K_0 = \int_0^{t_*/2} \sqrt{-q} dt. \quad (1.8)$$

In accordance with universal relation (4.5) from [4], the value of the constant c should be approximately the same for any polytropy indices k and pressure ratios ε . In fact, when $\varepsilon = 0$, $c = 0.546, 0.54$, and 0.536 for $k = 1.1, 1.4$, and 1.6 , i.e. the calculation confirms the existence of a universal relation $\tilde{H}(\bar{n})$. The approximation of H in the neighborhood of $\max H$ by a parabola

$$\overline{H} = H / \max H = 1 - c(n/n_0 - 1)^2, \quad c = 0,54 \quad (1.9)$$

satisfactorily describes H , as can be seen from Fig. 1 [lines 1 and 2 correspond to $\tilde{H}(\bar{n})$ [4] and (1.9)]. Using (1.9) and replacing sum (1.6) by an integral for $n \gg 1$, we can use (1.7) to obtain

$$\begin{aligned} \overline{\xi^2} &= \frac{k'T'}{2\pi R^3} 4^{N_1} e^W \int_1^{n_0} (\cos(2\sqrt{n} I_0))^{2N_1} e^{-Wc(n/n_0 - 1)^2} \left(\frac{q(0)}{-q(t_N)} \right)^{1/2} \frac{dn}{n}, \\ W &= 4NH_{\max}, \quad N_1 = [N - 0], \quad 2N = 1, 2, \dots \end{aligned} \quad (1.10)$$

It is difficult to calculate (1.10) analytically for an arbitrary unlimited number of cycles N_1 . However, such a calculation can be performed for a small number of cycles N_1 — which is of particular interest. We will calculate the integral (1.10) on the basis of two asymptotic methods. First, we formally have $I_0 \rightarrow \infty$. We then pass to the limit $Wc \rightarrow \infty$. Such an approach has been substantiated for short waves with a large exponent ($n \gg 1$) when the second asymptote does not disturb the first asymptote. The below condition validating the given approach follows from this

$$Wc(\Delta n/n_0)^2 \ll 1, \quad I_0 \Delta n / \sqrt{n_0} = \pi. \quad (1.11)$$

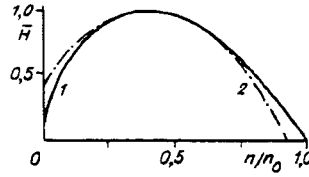


Fig. 1

The first formula of (1.11) stipulates that the exponent under the integral in (1.10) undergo a small change with a change in the wave exponent by a small amount ($\Delta n \ll n_0$). This corresponds locally to a period of variation $|\cos(2\sqrt{nI_0})|$, as reflected by Eq. (1.11). Using estimates that can be obtained from the formulas [3, 4]

$$c \max H \leq 0,25 \sqrt{n_0}, I_0 > \sqrt{\frac{3}{2}} \ln \frac{1}{R_*},$$

we find that (1.11) imposes the following limitation on the number of cycles

$$N_1 \ll \sqrt{n_0}. \quad (1.12)$$

It should be noted that N_1 is the number of cycles preceding the current cycle. Thus, $N = N_1 + 1$ for the second point of the cycle and $N = N_1 + 1/2$ for the first point. Since $N_1 = 0$, there are no limitations for the first cycle $N = 1$. With condition (1.12), we can replace the contribution of \cos under the integral (1.10) by its mean value and introduce the coefficient

$$e^{\Psi N_1} = \langle 4^{N_1} \cos^{2N_1}(2\sqrt{n}I_0) \rangle = \frac{4^{N_1}}{\pi} \int_0^\pi \cos^{2N_1} \varphi d\varphi, \quad (1.13)$$

$$e^{\Psi N_1} = 2, 6, 20, \dots, \quad \Psi = 0,69; 0,9; 1; \dots$$

Along with the above formulas, we also introduce the relation

$$R^{-3}(-q(t_n))^{1/2} \geq R_m^{-3}(-q(T/2))^{1/2}, \quad (1.14)$$

in which we have an equality for the first (or mean) point of the cycle and an inequality of the same order of magnitude for the second point. The inequality sign $>$ is important for obtaining a correct lower bound for $\bar{\xi}^2$. Passing in (1.10) to the limit $I_0 \rightarrow \infty$ and then $Wc \rightarrow \infty$ and using (1.13-1.14), we find that

$$\bar{\xi}^2 = \frac{k'T'}{2\sqrt{\pi c \sigma'}} \left(\frac{q(0)}{-q(T/2)} \right)^{1/2} \frac{\exp(W + \Psi N_1)}{R_m^3 \sqrt{W}}. \quad (1.15)$$

The formula for the mean square deviation of the surface (1.15) is valid for a slightly disturbed surface, while the characteristic deviation of the surface $\sqrt{\bar{\xi}^2}$ is much smaller than the wavelength associated with maximum growth of the disturbance Λ . When these quantities are of the same order of magnitude, there is a transition from a linear growth stage to a nonlinear stage. For an initially smooth bubble surface, the rupture condition corresponding to the end of the linear stage of growth can be taken in the form:

$$\sqrt{\bar{\xi}^2} = \frac{1}{4} \Lambda, \quad \Lambda = \frac{2\pi}{n_0} R', \quad R' = R_m R'_0. \quad (1.16)$$

To within a factor on the order of two, the form of rupture condition (1.16) is unimportant for calculations of bubble lifetime (time to rupture), since the number of pulsations corresponding to rupture N depends logarithmically on this factor.

The number of cycles to rupture N is found from Eqs. (1.15-1.16) with allowance for the formulas for n_0 , K_0 , H_{\max} (4.2) and (5.1) [4], Eqs. (1.7), (1.9), (1.10), and (1.13), and the values of k' , absolute temperature T' , surface tension σ' , initial radius R'_0 and pressure p_0 , the pressure gradient $1/\varepsilon$, and the polytropy index k .

2. Duration of Bubble Pulsations to Rupture of the Surface. Formulas (1.10), (1.15) for the square of the deviation of radius are valid for discrete values — an integral number $2N$ of half-lengths of the interval of exponential growth. As far as representation of the results is concerned, it is most convenient to approximately generalize the formulas to a continuous number of cycles and in so doing qualitatively account for the possibility of rupture at an arbitrary point of the interval.

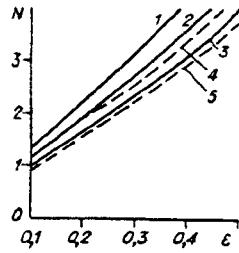


Fig. 2

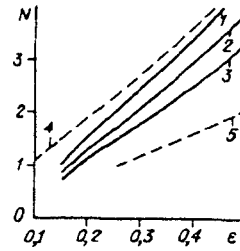


Fig. 3

Figure 2 shows results of calculations performed with (1.15-1.16) to establish the dependence of the number of cycles to rupture N on ε . The data is shown for different k and R_0' with $\sigma' = 0.072$ N/m (corresponding to water) and $p_0 = 0.1$ MPa. Viscosity was not considered in constructing the graphs in order to simplify representation of the data. The difference in the curves due to viscosity cannot be any more than 25% of the indicated values in the case of pure water (without substances added) at room temperature. Lines 4 and 5 correspond to $k = 1.66$ (helium) and $k = 1.31$ (methane or carbon dioxide) at $R_0' = 0.2$ cm. It is important that the number of pulsations to rupture is not large. When $\varepsilon = 0.1$, one cycle of pulsation is sufficient for extremely small initial perturbations caused by thermal fluctuations to grow to finite dimensions. This is half of the rupturing number of cycles estimated in [3]. It is evident from Fig. 2 that an increase in k from 1.4 to 1.66 affects N in roughly the same way as a decrease in R_0' from 0.2 to 0.1 cm. Lines 1-3 are given for $R_0' = 0.05, 0.1,$ and 0.2 cm when $k = 1.4$ (air or hydrogen).

It is also interesting to evaluate the time to rupture the surface of a vapor bubble by examining a similar gas bubble as an approximation. This approach is valid if the Florschuetz—Chao number [5] is small. The difference in the rates of phase transformation at the interface for gas bubbles and vapor bubbles can be characterized by a parameter analogous to the Florschuetz—Chao parameter [5]:

$$\Phi' = \frac{\rho_f \Delta T'}{\rho_g l} \frac{3}{R'} \sqrt{\kappa_f \tau}.$$

Here, $\Delta T'$ is the change in the temperature of the surface along the saturation curve; c_f and κ_f are the heat capacity and diffusivity of the fluid; l is the heat of phase transformation; ρ_g is the density of the gas. While $\Phi_0' < 1$ at the initial moment of time when radius is large ($R' \sim R_0'$), at $R_0' \rightarrow 0$ $\Phi' \rightarrow 0$ due to the decrease in $\tau \sim R^{5/2}$ and $1/\rho_g \sim R^3$. In this case, the bubbles will be close to gas bubbles when they collapse. Condensation will be negligible for small radii ($R \sim R_m$), since $\Phi' \ll 1$. Thus, when $R' \ll R_0'$, the vapor bubbles will be equivalent to gas bubbles in terms of the dynamics of the boundary. However, by virtue of the reduced mass of gas inside the bubble relative to the initial amount (a reduction by a factor of approximately $1 - \Phi'$), the actual values of effective initial gas pressure are markedly lower than p_0 . The effective initial temperature also undergoes a change. The effective pressure gradient $1/\varepsilon$ is considerably greater than the ratio p_∞/p_0 . Thus, calculations performed using the gas-bubble model with $\varepsilon = p_0/p_\infty$ should provide an upper bound for the number of cycles to rupture — which is of definite interest. Figure 3 shows results of calculations of the number of cycles to the rupture of a gas bubble having parameters similar to vapor bubbles in water ($k = 1.32$) when $R_0' = 0.05, 0.1,$ and 0.2 cm (curves 1-3). Line 4 shows the variant $R_0' = 0.05$ cm, $\sigma' = 0.056$ N/m, while line 5 shows the estimate for freon-113 ($k = 1.08$, $\sigma' = 0.015$ N/m, $T_c = 47.6^\circ\text{C}$, $p_0 = 0.1$ MPa). No calculation was performed for $\varepsilon < 0.3$ due to the proximity of the critical point of freon, while the calculations for the vapor bubbles were ended at $\varepsilon = 0.15$.

In the calculations performed for bubbles in water, we considered the change in surface tension with pressure in the bubble $\sigma'(p)$ along the saturation curve (lines 1-3 in Fig. 3). This was done using well-known tabulated data on $\sigma'(p)$. We took the mean pressure within the interval of exponential wave growth for the value of p . It is evident from Fig. 3 that changes in surface tension can have a significant effect on rupture. At $\varepsilon = 0.15$, the difference from the case $\sigma' = \text{const}$ reaches a factor of 1.5. The number of cycles to rupture is appreciably greater for the vapor bubbles than for freon, as indicated by the calculations performed using the gas bubble model.

In conclusion, we should emphasize the importance of allowing for fluid viscosity in the general case. Fluid viscosity can be accounted for by means of the formulas presented in [4]. To illustrate the importance of this, we note that a fivefold increase in the viscosity of water due to the addition of glycerin doubles the number of cycles to rupture compared to Fig. 2.

REFERENCES

1. M. L. Levin and S. M. Rytov, Theory of Equilibrium Fluctuations in Electrodynamics [in Russian], Nauka, Moscow (1967).
2. L. I. Mandel'shtam, Complete Collected Works, Vol. 1, Izd. An SSSR, Moscow (1948).
3. O. V. Voinov and V. V. Perepelkin, "Stability of the surface of a gas bubble pulsating in a liquid," Prikl. Mekh. Tekh. Fiz., No. 3 (1989).
4. O. V. Voinov, "Dynamics of capillary waves on a bubble with nonlinear pulsations in a low-viscosity fluid," Prikl. Mekh. Tekh. Fiz., No. 3 (1994).
5. L. W. Florschuetz and B. T. Chao, "On the mechanics of vapor bubble collapse," J. Heat Transfer, C87, No. 2 (1965).